Quantum Geometry, Inflation and a Small Cosmological Constant

Martin Bojowald

Center for Gravitational Physics and Geometry,
The Pennsylvania State University,
104 Davey Lab, University Park, PA 16802, USA

Abstract

Inflation is shown to be a natural consequence of quantum geometry, a canonical quantization of gravity. An inflaton is not required, but can be coupled resulting in large initial values. The mechanism is a modified density a^{-3} which results from quantum geometry.

Quantum geometry (loop quantum gravity) is a canonical quantization of gravity based on Ashtekar's variables. In contrast to the old Wheeler–DeWitt quantization it has a mathematically well-defined structure and predicts that the geometry of space and time is discrete.

One example, the spatial volume spectrum, is in the isotropic case of loop quantum cosmology

$$V_{\frac{1}{2}(|n|-1)} = 6^{-\frac{3}{2}} l_{\mathrm{P}}^3 \sqrt{(|n|-1)|n|(|n|+1)}$$

where $n \in \mathbb{Z}$ is an integer label of the eigenvalues and l_{P} is the Planck length.

That also time is discrete can be seen from the evolution equation for the wave function $s_n(\phi)$ (replacing $\psi(a,\phi)$ of the Wheeler–DeWitt quantization) which is a difference equation:

$$\begin{aligned}(V_{\frac{1}{2}|n+4|} - V_{\frac{1}{2}|n+4|-1}) s_{n+4}(\phi) - 2(V_{\frac{1}{2}|n|} - V_{\frac{1}{2}|n|-1}) s_n(\phi)\\ +(V_{\frac{1}{2}|n-4|} - V_{\frac{1}{2}|n-4|-1}) s_{n-4}(\phi) = -\tfrac{1}{3}\kappa l_{\mathrm{P}}^2 \hat{\mathcal{H}}_\phi(n) s_n(\phi)\end{aligned}$$

where $\kappa = 8\pi G$ is the gravitational constant and $\hat{\mathcal{H}}_\phi(n)$ the matter Hamiltonian for a field ϕ.

Instead of the usual internal time a we have the discrete time n whose norm is the eigenvalue of the operator $6\hat{a}^2/l_{\mathrm{P}}^2$. At large volume (large n) the Wheeler–DeWitt equation

$$\tfrac{1}{6} l_{\mathrm{P}}^4 a^{-1} \frac{\partial}{\partial a}\left(a^{-1}\frac{\partial}{\partial a}(a\psi(a,\phi))\right) = -\kappa \hat{\mathcal{H}}_\phi(a)\psi(a,\phi)$$

is recovered (in this ordering) as a continuum approximation.

At small volume the discreteness is essential and leads to a removal of the classical singularity at $a=0$ as well as to a modified cosmological evolution.

The Wheeler–DeWitt equation quantizes the Friedmann equation

$$H^2 = (\dot{a}/a)^2 = \tfrac{2}{3}\kappa a^{-3}\mathcal{H}_\phi(a)$$

with the matter Hamiltonian $\mathcal{H}_\phi(a)$, e.g. $\mathcal{H}_\phi(a) = \frac{1}{2}a^{-3}p_\phi^2 + a^3V(\phi)$ for a scalar ϕ (not necessarily an inflaton) with momentum p_ϕ. It turns out that the density $d(a) := a^{-3}$, which is responsible for the classical singularity, becomes discrete and modified at small a by quantum geometry effects.

One obtains an effective density which equals a^{-3} for large a, but is bounded and becomes, e.g.,

$$d_j(a) \simeq \tfrac{12^6}{7^6}(\tfrac{1}{3}l_{\mathrm{P}}^2 j)^{-15/2}a^{12} \quad \text{if} \quad a^2 \ll \tfrac{1}{3}l_{\mathrm{P}}^2 j$$

where $j \in \frac{1}{2}\mathbb{N}$ labels a quantization ambiguity. The effective density increases with a until $a^2 \simeq \frac{1}{3}l_{\mathrm{P}}^2 j$ where the classical behavior is approached (see Fig. 1). Thus, quantum geometry yields the effective Friedmann equation

$$H^2 = (\dot{a}/a)^2 = \tfrac{2}{3}\kappa a^{-3}(\tfrac{1}{2}d_j(a)p_\phi^2 + a^3V(\phi))$$

which we will study now.

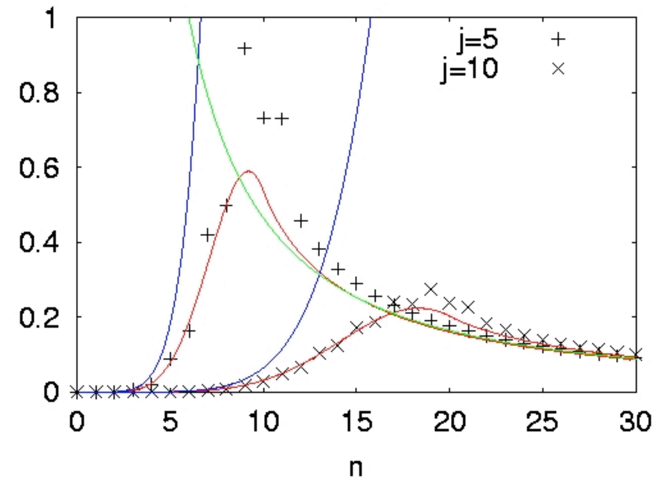

Figure 1: The effective density d_j (in Planck units; $n = 6a^2/l_{\mathrm{P}}^2$) for two values of j: discrete eigenvalues of the operator (+ and ×), the continuum approximation $d_j(a)$, the small-a approximations and the classical density a^{-3}.

From quantum geometry we derived the effective Friedmann equation

$$H^2 = (\dot{a}/a)^2 = \tfrac{2}{3}\kappa a^{-3}\left(\tfrac{1}{2}d_j(a)p_\phi^2 + a^3V(\phi)\right)$$

where the effective density $d_j(a)$ is increasing for small a. In standard, potential-driven inflation one has to arrange the evolution of the scalar in such a way that the potential term dominates over the kinetic term resulting in a right hand side of the Friedmann equation which does not decrease with a. This implies an accelerated expansion.

In our effective Friedmann equation, on the other hand, both the kinetic term and the potential term increase with a if a is small enough. Then, even for a vanishing potential we have the super-inflationary expansion (Fig. 2)

$$a(t) \propto (t_0 - t)^{-\frac{2}{9}} = (t_0 - t)^{\frac{2}{3(1+w)}} \quad \text{if} \quad a^2 \ll \tfrac{1}{3}l_{\mathrm{P}}^2 j$$

with $w = -4$ at very small a where $d_j(a) \propto a^{12}$ is increasing. For larger a, $d_j(a)$ grows less strongly with a such that w decreases until the maximum of $d_j(a)$ is reached. At this point, inflation stops and the universe exits gracefully into a standard phase.

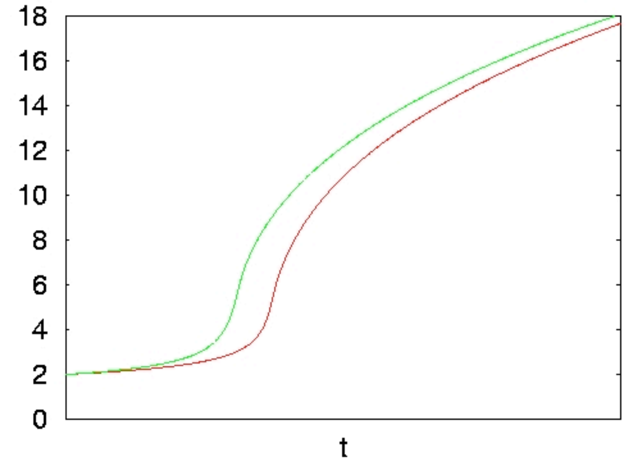

Figure 2: Solutions $a(t)$ (in Planck units) to the effective Friedmann equation with vanishing potential and a small quadratic potential, both with $j = 100$.

To know the amount of inflation we have to find out when inflation starts. However, this will be inside the Planck regime where we can no longer trust the effective equation for $a(t)$.

While the increasing behavior of the density $d_j(a)$ is true for all small a down to $a = 0$, it is not obvious from present techniques in which sense this would correspond to inflation. Rather than being described by a time evolution equation for $a(t)$, the universe evolves quantum mechanically, i.e. by the difference equation for our wave function $s_n(\phi)$. If the quantum evolution does correspond to inflation for all small values of a, as the density suggests, the number of e-foldings is certainly sufficient since $a(t_f)/a(t_i)$ would be arbitrarily large thanks to an arbitrarily small initial $a(t_i)$. In fact, the wave function shows the characteristic de Sitter behavior at all small a ($n = 6a^2/l_\mathrm{P}^2 < 400$ in Fig. 3), but the Planck regime has to be better understood.

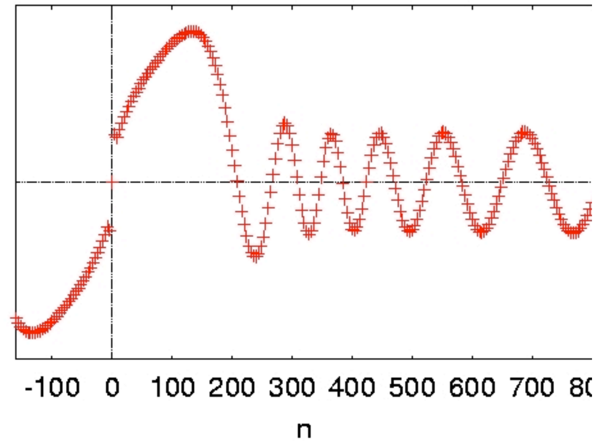

Figure 3: Solution s_n to the discrete time evolution equation ($j = 200$; negative n correspond to time before the classical singularity which is absent in the quantum description).

Quantum geometry predicts new qualitative features in the cosmological context. Quantitatively, however, they are affected by quantization ambiguities like j. Also the exponent l in the effective density $d_j(a) \propto a^l$ for small a is not unique, which affects the equation of state parameter w at early stages. It will, however, always be positive which implies inflation; in most cases it is even larger, $l > 3$, which means super-inflation.

This scenario can be combined with standard inflation: the field ϕ will increase during the phase of the modified density. Thus, if we choose it to be the inflaton, it will acquire a large initial value for a second phase of potential-driven inflation.

One can choose the parameter j for one matter component so large that the corresponding density $d_j(a)$ is still growing at present values of a. While this is less natural, it leads to "phantom matter" with equation of state parameter $w < -1$ which could explain the small value of today's cosmological constant. In this case, the effective Friedmann equation is

$$H^2 \propto (a/\sqrt{j}l_{\mathrm{P}})^9(\sqrt{j}l_{\mathrm{P}})^{-6} < (\sqrt{j}l_{\mathrm{P}})^{-6} < a^{-6}$$

which is very small since we need a very large $j > a^2/l_{\mathrm{P}}^2$. In particular, the effective cosmological constant $\Lambda = H^2$ would be time dependent via the scale factor a.

References

[1] M. Bojowald, Isotropic Loop Quantum Cosmology, *Class. Quantum Grav.* 19 (2002) 2717–2741

[2] M. Bojowald, Absence of a Singularity in Loop Quantum Cosmology, *Phys. Rev. Lett.* 86 (2001) 5227–5230

[3] M. Bojowald, Dynamical Initial Conditions in Quantum Cosmology, *Phys. Rev. Lett.* 87 (2001) 121301

[4] M. Bojowald, Inflation from quantum geometry, gr-qc/0206054